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Parabolic equations with BMO coefficients in Lipschitz domains ☆

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Abstract

In this paper, we are concerned with certain natural Sobolev-type estimates for weak solutions of inhomogeneous problems for second-order parabolic equations in divergence form. The geometric setting is that of time-independent cylinders having a space intersection assumed to be locally given by graphs with small Lipschitz coefficients, the constants of the operator being uniformly parabolic. We prove the relevant L^p estimates, assuming that the coefficients are in parabolic bounded mean oscillation (BMO) and that their parabolic BMO semi-norms are small enough.

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Keywords: Parabolic equation; Lipschitz domain; Maximal function; Vitali covering lemma

1. Introduction

There have been a number of results concerning L^p estimates for parabolic equations in nondivergence form with discontinuous coefficients (see [6,22–26]). However, to the best of our knowledge no rigorous results concerning the regularizing properties of parabolic equations in divergence form with bounded mean oscillations (BMO) coefficients in nonsmooth domains are available in the literature for this class of problems. The study of parabolic equations closely parallels the study of elliptic equations.

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Recently, the author [7] has investigated suitable and minimal conditions on the coefficients and domains for the $W^{1,p}$ regularity theory for the divergence form elliptic equation. This work is a natural follow up to the article [7] in the parabolic setting. Our concern in this paper is primarily to develop a general theory of certain natural Sobolev-type estimates for weak solutions of the following initial/boundary value problem:

$$\begin{cases} u_t - (a_{ij}u_{x_j})_{x_i} = u_t - div(A\nabla u) = & div\,\mathbf{f} = (f^i)_{x_i} \quad \text{in} \quad \Omega_T, \\ \qquad\qquad\qquad\qquad\qquad\quad u = & 0 \qquad\qquad\quad \text{on} \quad \partial_p \Omega_T, \end{cases} \tag{1.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^n$, $\Omega_T = \Omega \times (0, T]$ is a cylinder in $\mathbb{R}^n \times \mathbb{R}$ and $\partial_p \Omega_T = \partial\Omega \times [0, T] \cup \Omega \times \{t = 0\}$ is the parabolic boundary of Ω_T.

We introduce an intrinsic Sobolev space $W_*^{1,p}$ (see Definition 2.4)

$$W_*^{1,p}(\Omega_T) = L^p\left(0, T; W^{1,p}(\Omega)\right) \cap W^{1,p}\left(0, T; W^{-1,q}(\Omega)\right)$$

and

$$\mathring{W}_*^{1,p}(\Omega_T) = \overline{C_0^\infty(\Omega_T)} \text{ in } W_*^{1,p}(\Omega_T),$$

where $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $W^{-1,q}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$. Roughly speaking, functions in $W_*^{1,p}(\Omega_T)$ have spatial derivatives in L^p and have time derivatives which can be written as the sum of an L^p function and the spatial derivative of an L^p function. The purpose of this paper is to show the well posedness in $\mathring{W}_*^{1,p}(\Omega_T)$ of the Dirichlet problem (1.1) with the estimate

$$\|u\|_{W_*^{1,p}(\Omega_T)} \leqslant C\|\mathbf{f}\|_{L^p(\Omega_T)} \tag{1.2}$$

for some constant C independent of u and $\mathbf{f}$. Our basic approach is similar to that developed in [27].

Throughout this paper, the coefficients of the operator are supposed to be defined on $\mathbb{R}^n \times \mathbb{R}$, as follows from [1,18]. The main assumptions on the coefficients are that they are in parabolic BMO and their parabolic BMO semi-norms are small enough. We use the following definition.

Definition 1.1. We say that the matrix A of coefficients is (δ, R)-vanishing if

$$\sup_{0<r\leqslant R} \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} \sqrt{\frac{1}{|C_r|}\int_{C_r(x,t)} \left|A(y,s) - \overline{A}_{C_r(x,t)}\right|^2 dy\,ds} \leqslant \delta, \tag{1.3}$$

where $C_r(x,t) = B_r(x,t) \times (t - r^2/2, t + r^2/2]$ is a centered parabolic cube and $\overline{A}_{C_r(x,t)}$ is the average of A over $C_r(x,t)$.

We would like to point out that our assumptions that A is (δ, R)-vanishing relax and hence generalize the results established in VMO (see e.g. [2,6,12,13,15,22–26]).

Our geometric setting in this paper is that of time-independent cylinders having a space intersection assumed to be locally given by graphs with small Lipschitz constants. More precisely, we have the following definition.

Definition 1.2. We say that a domain Ω is (δ, R)-Lipschitz if every $x_0 \in \partial\Omega$ and every $r \in (0, R]$, there exists a Lipschitz continuous function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega \cap B_r(x_0) = \left\{x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n) \in B_r(x_0) : x_n > \gamma(x')\right\}$$

and

$$\sup_{x', y' \in T_r(x_0), x' \neq y'} \frac{|\gamma(x') - \gamma(y')|}{|x' - y'|} \leqslant \delta$$

in some coordinate system.

We remark that γ in the definition above is Lipschitz continuous with small Lipschitz constant if and only if it is in $W^{1,\infty}$ with small $\|\nabla\gamma\|_{L^\infty}$ (see [16, Theorem 4 of Chapter 5]). For further discussions regarding some works on Lipschitz domains we refer to the papers (e.g. [3,4,17]). We would like to point out that our assumption that Ω is (δ, R)-Lipschitz weakens the assumption in [15] that $\partial\Omega$ is in $C^{1,1}$ and the assumption in [2] that $\partial\Omega$ is in C^1. We remark that one might assume that R in both Definitions 1.1 and 1.2 is 1 by scaling the given equations, while δ is scaling invariant. In this paper, we mean δ to be a small positive constant and we want to establish the estimate (1.2) under the assumptions that A is (δ, R)-vanishing and Ω is (δ, R)-Lipschitz.

According to classical works when $p = 2$ (see [5,19–21]), as long as A is uniformly parabolic (see Definition 2.1) and $\mathbf{f} \in L^2(\Omega_T)$, (1.1) has a unique weak solution u; that is, u is a function in

$$C^0(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$$

satisfying the integral identity

$$\int_{\Omega_T} u\varphi_t \, dx \, dt - \int_{\Omega_T} A\nabla u \nabla\varphi \, dx \, dt = \int_{\Omega_T} \mathbf{f}\nabla\varphi \, dx \, dt.$$

In addition this solution belongs to

$$L^2(0, T; H_0^1(\Omega)) \cap H^{1/2}(0, T; L^2(\Omega)).$$

We recall some of them in the next section. We would like to remark that

$$\mathring{W}_*^{1,2} = L^2(0,T;H_0^1(\Omega)) \cap H^{1/2}(0,T;L^2(\Omega)).$$

Consequently, the estimate (1.2) holds true under the assumptions considered in this work when $p = 2$. We will hereafter focus attention exclusively on the case that $p > 2$. The case $1 < p < \infty$ will be easily recovered by a duality argument.

We now come to state the definition of our weak solutions.

Definition 1.3. Let $1 < p,\ q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then a weak solution of (1.1) is a function $u \in \mathring{W}_*^{1,p}(\Omega_T)$ such that

$$\int_{\Omega_T} u\varphi_t \, dx\, dt - \int_{\Omega_T} A\nabla u \nabla \varphi \, dx\, dt = \int_{\Omega_T} \mathbf{f} \nabla \varphi \, dx\, dt$$

for all $\varphi \in \mathring{W}_*^{1,q}(\Omega_T)$ with $\varphi = 0$ for $t = T$.

Remark 1.4. We remark that by an approximation argument, we can take φ with $\varphi = 0$ for $t = T$ from the space $C_0^\infty(\Omega_T)$.

Let us state the main result of this work.

Theorem 1.5. *Given $p > 1$, there is a small $\delta = \delta(\Lambda, p, n, R) > 0$ so that for all A with A uniformly parabolic (see Definition 2.1) and (δ, R)-vanishing, for all Ω with Ω (δ, R)-Lipschitz, and for all $\mathbf{f}$ with $\mathbf{f} \in L^p(\Omega_T; \mathbb{R}^n)$, the Dirichlet problem (1.1) has a unique weak solution u with the estimate*

$$\|u\|_{W_*^{1,p}(\Omega_T)} \leqslant C \|\mathbf{f}\|_{L^p(\Omega_T)},$$

where the constant C is independent of u and $\mathbf{f}$.

Our approach is carried out by an argument of approximation, which is based on the parabolic maximal function, Vitali covering lemma, good Λ-inequalities, and energy estimates (see e.g. [7–9,11]). Our approach is very much influenced by [11,27]. In [11], the Calderón-Zygmund decomposition was used to obtain interior L^p estimates. We use the Vitali covering lemma to obtain global L^p estimates as in [7–9]. Our basic tools in this approach are the Vitali covering lemma, the Hardy–Littlewood maximal function and the compactness method.

This paper will be organized as follows: In Section 2, we will record auxiliary notations, relevant function spaces, some definitions and some geometric analysis results related to our approach. In Section 3, we discuss interior estimates. Boundary estimates will be obtained for the Dirichlet problem (1.1) in Section 4. Our optimal regularity requirements on the coefficients and the domain will be discussed with the proof of our global $\mathring{W}_*^{1,p}$ estimates in Section 5.

2. Some preliminary facts from real analysis

2.1. Geometric notation

(1) $\mathbb{R}^n = n$-dimensional real Euclidean space.
(2) $e_i = (0, \ldots, 1, \ldots, 0) = i$th standard coordinate vector.
(3) A typical point in $\mathbb{R}^n \times \mathbb{R}$ is $(x, t) = (x', x_n, t)$.
(4) $R^n_+ = \{x \in R^n : x_n > 0\}$.
(5) $B_r = \{y \in \mathbb{R}^n : |y| < r\}$ is an open ball on R^n with center 0 and radius $r > 0$, $B_r(x) = B_r + x$, $B_r^+ = B_r \cap \mathbb{R}^n_+$, $B_r^+(x) = B_r^+ + x$, $T_r = B_r \cap \{x_n = 0\}$, $T_r(x) = T_r + x$, and $\partial_c B_r^+ = \partial B_r \cap \mathbb{R}^n_+$ is the curved part of ∂B_r^+.
(6) $\Omega_T = \Omega \times (a, a+T]$ $(a > 0)$ is a parabolic cylinder, $S\Omega_T = \partial\Omega \times [a, a+T]$ is the lateral boundary of Ω_T, and $\partial_p \Omega_T = S\Omega_T \cup \Omega \times \{a\}$ is its parabolic boundary.
(7) $Q_r = B_r \times (-r^2, 0]$ is a parabolic cube, $Q_r(x,t) = Q_r + (x,t)$, $\partial_p Q_r = \partial B_r \times [-r^2, 0] \cup B_r \times \{-r^2\}$ is its parabolic boundary, $Q_r^+ = B_r^+ \times (-r^2, 0]$, $Q_r^+(x,t) = Q_r^+ + (x,t)$, $\hat{T}_r = T_r \times [-r^2, 0]$, and $\hat{T}_r(x,t) = \hat{T}_r + (x,t)$.
(8) $C_r = B_r \times (r^2/2, r^2/2]$ is a centered parabolic cube, $C_r(x,t) = C_r + (x,t)$.

2.2. Matrix of coefficients

(1) We write $A = \{a_{ij}\}$ to mean an $n \times n$ matrix with (i, j)th entry a_{ij}.
(2) $|A| = \sqrt{(A : A)} = \sqrt{\sum_{i,j=1}^n a_{ij}^2}$ and $\|A\|_\infty = \sup_{(y,s)} |A(y,s)|$.
(3) A is supposed to be uniformly parabolic (see Definition 2.1 below).
(4) A is supposed to be (δ, R)-vanishing (see Definition 1.1).
(5) In this paper A is allowed to be nonsymmetric.

Definition 2.1. We say that the matrix of coefficients A is uniformly parabolic if there exists a positive constant Λ such that

$$\Lambda^{-1}|\xi|^2 \leqslant A(x,t)\xi \cdot \xi \leqslant \Lambda|\xi|^2, \quad \text{a.e. } (x,t) \in \mathbb{R}^n \times \mathbb{R} \quad \forall \xi \in \mathbb{R}^n.$$

2.3. Notation for function

(1) If $u : \Omega_T \to \mathbb{R}$, we write $u(x,t)$ $((x,t) \in \Omega_T)$. If $\mathbf{f} : \Omega_T \to \mathbb{R}^n$, we write $\mathbf{f}(x,t) = (f^1(x,t), \ldots, f^n(x,t))$.
(2)
$$\overline{f}_{C_r} = \frac{1}{|C_r|} \int_{C_r} f(x,t)\, dx\, dt$$
is the average of f over C_r.

2.4. Notation for derivatives

(1) $\nabla u = (u_{x_1}, \ldots, u_{x_n})$ is the gradient of u with respect to spatial variable x.
(2) $div\, \mathbf{f} = \sum_{i=1}^n (f^i)_{x_i} = (f^i)_{x_i}$ is the divergence of $\mathbf{f} = (f^1, f^2, \ldots, f^n)$.

2.5. Notation for estimates

We employ the letter C to denote a universal constant depending usually on the dimension, uniform parabolicity, and the geometric quantities of $\partial_p \Omega_T$.

2.6. Function spaces

(1) The Sobolev space $W_p^{1,0}(\Omega_T)$ $(1 < p < \infty)$ is the Banach space consisting of all elements of $L^p(\Omega_T)$ having a finite norm

$$\|u\|_{W_p^{1,0}(\Omega_T)} = (\|u\|^p_{L^p(\Omega_T)} + \|\nabla u\|^p_{L^p(\Omega_T)})^{1/p}.$$

If $p = 2$, we usually write $H^{1,0}(\Omega_T) = W_2^{1,0}(\Omega_T)$. The letter H is used, since—as we see—$H^{1,0}(\Omega_T)$ is a Hilbert space with scalar product

$$(u, v)_{H^{1,0}(\Omega_T)} = \int_{\Omega_T} (uv + \nabla u \nabla v)\, dx\, dt.$$

(2) The Sobolev space $W_p^{1,1}(\Omega_T)$ $(1 < p < \infty)$ is the Banach space consisting of all elements of $L^p(\Omega_T)$ having a finite norm

$$\|u\|_{W_p^{1,1}(\Omega_T)} = (\|u\|^p_{L^p(\Omega_T)} + \|u_t\|^p_{L^p(\Omega_T)} + \|\nabla u\|^p_{L^p(\Omega_T)})^{1/p}.$$

If $p = 2$, we write $H^{1,1}(\Omega_T) = W_2^{1,1}(\Omega_T)$. The letter H is used, since—as we see—$H^{1,1}(\Omega_T)$ is a Hilbert space with scalar product

$$(u, v)_{H^{1,1}(\Omega_T)} = \int_{\Omega_T} (uv + \nabla u \nabla v + u_t v_t)\, dx\, dt.$$

(3) The Sobolev space $W_\infty^{1,1}(\Omega_T)$ is the Banach space consisting of all elements of $L^p(\Omega_T)$ having a finite norm

$$\|u\|_{W_\infty^{1,1}(\Omega_T)} = ess \sup_{\Omega_T} |u| + ess \sup_{\Omega_T} |\nabla u| + ess \sup_{\Omega_T} |u_t|.$$

Weak solutions considered hereafter are supposed to be defined on $\Omega \times \mathbb{R}$, as follows from the fact that the solution u and the equation can be extended beforehand onto the larger cylinder $\Omega_{T+\varepsilon}$ with preservation of all properties of the functions in question along with our zero parabolic boundary condition. Now, we introduce a certain non-isotropic Sobolev space whose members have weak derivatives of spatial order 1 and time order $\frac{1}{2}$ lying in the L^2 spaces. For this space, we refer to the book [19].

The space $H^{1,1/2}(\Omega\times\mathbb{R})$ consists of all elements u of $H^{1,0}(\Omega\times\mathbb{R})$ having a finite integral

$$\|D^t_{1/2}\|_{L^2(\Omega\times\mathbb{R})}=\sqrt{\int_{\mathbb{R}}\left(\frac{\|u(x,t+h)-u(x,t)\|_{L^2(\Omega\times\mathbb{R}}}{h^{1/2}}\right)^2\frac{dh}{h}}.$$

Remark 2.2 (*Ladyzhenskaya et al.* [19]). $H^{1,1/2}(\Omega\times\mathbb{R})$ is a Hilbert space if its norm is defined by the equality

$$\|u\|_{H^{1,1/2}(\Omega\times\mathbb{R})}=\sqrt{\|u\|^2_{L^2(\Omega\times\mathbb{R})}+\|\nabla u\|^2_{L^2(\Omega\times\mathbb{R})}+\|D^t_{1/2}\|^2_{L^2((\Omega\times\mathbb{R})}}.$$

Remark 2.3 (*Ladyzhenskaya et al.* [19]).

$$\|D^t_{1/2}\|^2_{L^2(\Omega\times\mathbb{R})}=\int_{\mathbb{R}}\int_{\Omega}|\widehat{u}(x,s)|^2\,|s|\,dx\,ds,$$

where $\widehat{u}(x,s)$ is the Fourier transform of $u(x,t)$ with respect to t; that is,

$$\widehat{u}(x,s)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}u(x,t)e^{-ist}\,dt.$$

The space we mainly deal with in this paper is $W^{1,p}_*(\Omega_T)$ mentioned in the Introduction.

Definition 2.4. We say $u\in W^{1,p}_*(\Omega_T)$ $(1<p<\infty)$ if $u\in W^{1,0}_p(\Omega_T)$ and there exist functions $\mathbf{F}\in L^p(\Omega_T;\mathbb{R}^n)$ and $g\in L^p(\Omega_T)$ such that

$$u_t=div\,\mathbf{F}-g$$

in Ω_T in the sense that

$$\int_{\Omega_T}u\varphi_t\,dx\,dt=\int_{\Omega_T}(\mathbf{F}\nabla\varphi+g\varphi)\,dx\,dt,\quad\forall\varphi\in C^\infty_0(\Omega_T).\tag{2.1}$$

Furthermore, we define its norm by

$$\|u\|_{W^{1,p}_*(\Omega_T)}=\|u\|_{W^{1,0}_p(\Omega_T)}+\inf\left\{\left(\int_{\Omega_T}(|\mathbf{F}|^p+|g|^p)\,dx\,dt\right)^{1/p}\right\},$$

where the infimum runs over all the functions satisfying (2.1). We denote by $\mathring{W}^{1,p}_*$ the closure of $C^\infty_0(\Omega_T)$ in $W^{1,p}_*(\Omega_T)$.

It follows from the above definition that

$$W_*^{1,p}(\Omega_T) = L^p\left(0,T;W^{1,p}(\Omega)\right) \cap W^{1,p}(0,T;W^{-1,q}(\Omega)),$$

where $1<p<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. In particular, for the classic case that $p=2$ (see [5,19]), we have

$$W_*^{1,2}(\Omega_T) = L^2\left(0,T;H^1(\Omega)\right) \cap H^{1/2}(0,T;L^2(\Omega)).$$

2.7. Preliminary tools

We use the parabolic maximal function. The parabolic maximal function is defined as

$$(\mathcal{M}f)(x,t) = \sup_{r>0} \frac{1}{|C_r(x,t)|}\int_{C_r(x,t)} |f(y,s)|\,dy\,ds,$$

which satisfies strong $p-p$ estimates and weak $1-1$ estimates (see [27, Theorem 2] or [14, Section 1 of Chapter 2]). We also use

$$\mathcal{M}_{\Omega_T} f = \mathcal{M}(\chi_{\Omega_T} f)$$

if f is not defined outside Ω_T, where χ_{Ω_T} is the indicator function of Ω_T. We will drop the index Ω_T if Ω_T is understood clearly in the context.

The main technical tool for interior estimates is the following version of the Vitali covering lemma.

Lemma 2.5 (*Wang [27]*). *Let $0<\varepsilon<1$ and let $E\subset F\subset Q_1$ be two measurable sets with*

$$|E| < \varepsilon |C_1|$$

and satisfying the following property:

$$\textit{for every } (x,t)\in Q_1 \textit{ with } |E\cap C_r(x,t)| \geqslant \varepsilon |C_r|,\;\; C_r(x,t)\cap Q_1 \subset F.$$

Then

$$|E| \leqslant (10)^{n+2}\varepsilon |F|.$$

We need another version of the Vitali covering lemma for boundary estimates.

Theorem 2.6. *Let $0 < \varepsilon < 1$ and let $E \subset F \subset Q_1^+$ be two measurable sets with*

$$|E| < \varepsilon \left|Q_1^+\right|. \tag{2.2}$$

Assume that the following property holds:

$$\forall\,(x,t) \in Q_1^+ \ \forall r \in (0,1] \text{ with } |E \cap C_r(x,t)| \geqslant \varepsilon |C_r(x,t)|, \quad C_r(x,t) \cap \Omega \subset F. \tag{2.3}$$

Then

$$|E| \leqslant 2(10)^{n+2} \varepsilon |F|.$$

Proof. From (2.2), it follows that for almost every $(x,t) \in E$, there exists a small $r_{(x,t)} > 0$ such that

$$|E \cap C_{r_{(x,t)}}(x,t)| = \varepsilon |C_{r_{(x,t)}}| \quad \text{and} \quad |E \cap C_r(x,t)| < \varepsilon |C_r(x,t)| \tag{2.4}$$

for all $r \in \left(r_{(x,t)}, 1\right]$. By the Vitali covering lemma, there exists a disjoint covering $\left\{C_{r_i}(x_i,t_i) \cap E : (x_i,t_i) \in E\right\}_{i=1}^{\infty}$ such that

$$E \subset \bigcup_i C_{5r_i}(x_i,t_i) \quad \text{and} \quad |E| \leqslant 5^{n+2} \sum \left|C_{r_i}\right|. \tag{2.5}$$

Then from (2.4), it follows that

$$\left|E \cap C_{5r_i}(x_i,t_i)\right| < \varepsilon \left|C_{5r_i}\right| = 5^{n+2} \varepsilon \left|C_{r_i}\right| = 5^{n+2} \left|E \cap C_{r_i}(x_i,t_i)\right|. \tag{2.6}$$

Observing $r_i \leqslant 1$, we claim that

$$\left|C_{r_i}\right| \leqslant 2^{n+3} \left|C_{r_i}(x_i,t_i) \cap Q_1^+\right|. \tag{2.7}$$

For any fixed $r > 0$, we observe that

$$\inf_{(x,t) \in Q_1^+} \{|C_r(x,t) \cap Q_1^+|\} = |C_r(e_1,0) \cap Q_1^+|$$

and

$$C_r(e_1,0) \cap Q_1^+ \supset C^+_{\frac{r}{2}}\left((1-\frac{r}{2})(e_1,0)\right).$$

Then we calculate

$$\left|C_r(x,t)\cap Q_1^+\right| \geqslant \left|C_r(e_1,0)\cap Q_1^+\right| \geqslant \left|C_{\frac{r}{2}}^+\right| = 2^{-(n+3)}\left|C_r(x,t)\right|.$$

This establishes (2.7).

In light of (2.5)–(2.7) and (2.3), we thus deduce

$$\begin{aligned}
|E| &= \left|\bigcup_i \left(C_{5r_i}(x_i,t_i)\cap E\right)\right| \\
&\leqslant \sum_i \left|C_{5r_i}(x_i,t_i)\cap E\right| \\
&< \varepsilon \sum_i \left|C_{5r_i}(x_i,t_i)\right| \\
&= 5^{n+2}\varepsilon \sum_i \left|C_{r_i}(x_i,t_i)\right| \\
&\leqslant \left(5^{n+2}\varepsilon\right)\left(2^{n+3}\right)\sum_i \left|\left(C_{r_i}(x_i,t_i)\cap Q_1^+\right)\right| \\
&= 2\,(10)^{n+2}\,\varepsilon\left|\bigcup_i \left(C_{r_i}(x_i,t_i)\cap Q_1^+\right)\right| \\
&\leqslant 2\,(10)^{n+2}\,\varepsilon |F|
\end{aligned}$$

and this completes our proof. □

3. Interior estimates

This section develops interior $W_*^{1,p}$-regularity theory concerning the following divergence form parabolic equation:

$$u_t - div(A\nabla u) = div\,\mathbf{f} \tag{3.1}$$

in a bounded parabolic cylinder in $\Omega_T = \Omega \times (a, a+T] \subset \mathbb{R}^n \times \mathbb{R}$, where $a > 0$. Our main assumption in this section is that the matrix $A(x,t)$ of coefficients is (δ, R)-vanishing; that is, the coefficients of the operator have small BMO semi-norms.

Definition 3.1. We say that $u \in W_*^{1,2}(Q_R)$ is a weak solution of (3.1) in Q_R if for all $\varphi \in C_0^\infty(Q_R)$ which vanish on $B_R \times \{0\}$, we have

$$\int_{Q_R} u\varphi_t \, dx\, dt - \int_{Q_R} A\nabla u \nabla \varphi \, dx\, dt = \int_{Q_R} \mathbf{f} \nabla \varphi \, dx\, dt.$$

Let us state the main result of this section.

Theorem 3.2. *Given $p > 2$, there is a small $\delta = \delta(\Lambda, p, n, R) > 0$ so that for all A with A uniformly parabolic and (δ, R)-vanishing, and for all $\mathbf{f}$ with $\mathbf{f} \in L^p\left(Q_7(0,2); \mathbb{R}^n\right)$, if u is a weak solution of the parabolic PDE* (3.1) *in $\Omega_T \supset Q_7(0,2)$, then $u \in W_*^{1,p}(Q_1)$ with the estimate*

$$\|u\|_{W_*^{1,p}(Q_1)} \leqslant C\left(\|u\|_{L^p(Q_7(0,2))} + \|\mathbf{f}\|_{L^p(Q_7(0,2))}\right),$$

where the constant C is independent of u and $\mathbf{f}$.

Remark 3.3. In view of a scaling argument, we can change the parabolic cube $Q_7(0,2)$ in Theorem 3.2 to any cube $Q_R(0, \varepsilon+)$ with $R > 1$.

We need the following standard energy estimates for the compactness argument and for the proof of Corollary 3.7. For their proofs see Lemmas 4.6, 4.8 and Theorem 4.7.

Lemma 3.4. *Let u be a weak solution of the parabolic PDE* (3.1) *in Q_2. Then we have*

$$\int_{Q_1} |\nabla u|^2 \, dx\, dt \leqslant C \int_{Q_2} \left(|u|^2 + |\mathbf{f}|^2\right) dx\, dt.$$

Lemma 3.5. *Let u be a weak solution of* (3.1) *in Q_2. Then we have*

$$\left\|u - \bar{u}_{Q_1}\right\|^2_{W_*^{1,2}(Q_1)} \leqslant C\left(\|\nabla u\|^2_{L^2(Q_1)} + \|\mathbf{f}\|^2_{L^2(Q_1)}\right). \tag{3.2}$$

We will use the following approximation lemma to study the deviation of the coefficients of the operator.

Lemma 3.6. *For any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that for any weak solution u of* (3.1) *in Q_4 with*

$$\frac{1}{|Q_4|} \int_{Q_4} |\nabla u|^2 \, dx\, dt \leqslant 1 \tag{3.3}$$

and

$$\frac{1}{|Q_4|}\int_{Q_4}\left(|\mathbf{f}|^2+\left|A-\overline{A}_{Q_4}\right|^2\right)dx\,dt\leqslant\delta^2, \tag{3.4}$$

there exists a weak solution v *of*

$$v_t-div\left(\overline{A}_{Q_4}\nabla v\right)=0$$

in Q_4 *such that*

$$\int_{Q_4}|(u-\overline{u}_{Q_4})-v|^2\,dx\,dt\leqslant\varepsilon^2. \tag{3.5}$$

Proof. We prove this by contradiction. If not, there exist $\varepsilon_0>0$, $\{A_k\}_{k=1}^{\infty}$, $\{u_k\}_{k=1}^{\infty}$ and $\{\mathbf{f}_k\}_{k=1}^{\infty}$ such that

$$(u_k)_t-div(A_k\nabla u_k)=div\,\mathbf{f}_k \tag{3.6}$$

in Q_4 and

$$\frac{1}{|Q_4|}\int_{Q_4}|\nabla u_k|^2\,dx\,dt\leqslant 1,\quad \frac{1}{|Q_4|}\int_{Q_4}(|\mathbf{f}_k|^2+|A_k-\overline{A_k}_{Q_4}|^2)\leqslant\frac{1}{k^2}. \tag{3.7}$$

But,

$$\int_{Q_4}\left|\left(u_k-\overline{u_k}_{Q_4}\right)-v_k\right|^2\,dx\,dt\geqslant\varepsilon_0^2. \tag{3.8}$$

for any weak solution v_k of

$$(v_k)_t-div\left(\overline{A_k}_{Q_4}\nabla v_k\right)=0\quad\text{ in } Q_4. \tag{3.9}$$

In view of (3.6), (3.7) and Lemma 3.5, $\left\{u_k-\overline{u_k}_{Q_4}\right\}_{k=1}^{\infty}$ is bounded in $W_*^{1,2}(Q_4)$. Consequently, it has a subsequence, which we still denote by $\left\{u_k-\overline{u_k}_{Q_4}\right\}$, such that

$$u_k-\overline{u_k}_{Q_4}\to u_0\quad\text{in}\quad L^2(Q_4),\ \ u_k-\overline{u_k}_{Q_4}\rightharpoonup u_0\quad\text{in}\quad W_*^{1,2}(Q_4). \tag{3.10}$$

For this compactness argument, we refer to the papers [5,19].

Since $\{\overline{A_k}_{Q_4}\}_{k=1}^{\infty}$ is bounded in ℓ^{∞}, it has a subsequence, which we still denote by $\{\overline{A_k}_{Q_4}\}_{k=1}^{\infty}$, such that

$$\left\| \overline{A_k}_{Q_4} - A_0 \right\|_{\infty} \to 0 \quad \text{as} \quad k \to \infty \tag{3.11}$$

for some constant matrix A_0. But then, by (3.7), we have

$$A_k \to A_0 \text{ in } L^2(Q_4). \tag{3.12}$$

We will now claim that u_0 is a weak solution of

$$(u_0)_t - div(A_0 \nabla u_0) = 0 \quad \text{in} \quad Q_4. \tag{3.13}$$

To prove this, choose any $\varphi \in C_0^{\infty}(Q_4)$ with $\varphi = 0$ for $t = 0$. From (3.7), we have

$$\int_{Q_4} \left(u_k - \overline{u_k}_{Q_4}\right) \varphi_t \, dx \, dt - \int_{Q_4} A_k \nabla u_k \nabla \varphi \, dx \, dt = \int_{Q_4} \mathbf{f}_k \nabla \varphi \, dx \, dt.$$

We recall (3.10), (3.11) and (3.7) to find upon passing to weak limits that

$$\int_{Q_4} u_0 \varphi_t \, dx \, dt - \int_{Q_4} A_0 \nabla u_0 \nabla \varphi \, dx \, dt = 0,$$

which shows (3.13).

We observe from (3.13) that

$$\begin{aligned} (u_0)_t - div\left(\overline{A_k}_{Q_4} \nabla u_0\right) &= (u_0)_t - div\left(\left[\overline{A_k}_{Q_4} - A_0\right] \nabla u_0\right) - div\left(A_0 \nabla u_0\right) \\ &= -div\left(\left[\overline{A_k}_{Q_4} - A_0\right] \nabla u_0\right) + (u_0)_t - div\left(A_0 \nabla u_0\right) \\ &= -div\left(\left[\overline{A_k}_{Q_4} - A_0\right] \nabla u_0\right) \end{aligned}$$

in O_4. Next, we let h_k be the weak solution of

$$\begin{cases} (h_k)_t - div(\overline{A_k}_{Q_4} \nabla h_k) = -div([\overline{A_k}_{Q_4} - \overline{A_0}] \nabla u_0) & \text{in} \quad Q_4, \\ h_k = 0 & \text{on} \quad \partial_p Q_4. \end{cases} \tag{3.14}$$

Then $u_0 - h_k$ is a weak solution of

$$(u_0 - h_k)_t - div\left(\overline{A_k}_{Q_4} \nabla (u_0 - h_k)\right) = 0 \tag{3.15}$$

in Q_4. From (3.14), we see

$$\begin{aligned}\|h_k\|_{L^2(Q_4)} &\leqslant \|h_k\|_{H^{1,1}(Q_4)}\\ &\leqslant C\|(\overline{A_k}_{Q_4} - A_0)\nabla u_0\|_{L^2(Q_4)}\\ &\leqslant C\|\overline{A_k}_{Q_4} - A_0\|_\infty \|\nabla u_0\|_{L^2(Q_4)}\\ &\leqslant C\|\overline{A_k}_{Q_4} - A_0\|_\infty.\end{aligned}$$

Consequently

$$\begin{aligned}\|(u_k - \overline{u_k}_{Q_4}) - (u_0 - h_k)\|_{L^2(Q_4)} &\leqslant \|(u_k - \overline{u_k}_{Q_4}) - u_0\|_{L^2(Q_4)} + \|h_k\|_{L^2(Q_4)}\\ &\leqslant \|(u_k - \overline{u_k}_{Q_4}) - u_0\|_{L^2(Q_4)}\\ &\quad + C\|\overline{A_k}_{Q_4} - \overline{A_0}\|_\infty.\end{aligned}$$

Then this estimate, (3.10) and (3.11) imply

$$\left\|\left(u_k - \overline{u_k}_{Q_4}\right) - (u_0 - h_k)\right\|_{L^2(Q_4)} \to 0 \quad \text{as } k \to \infty.$$

But this is a contradiction to (3.8) by (3.15). $\square$

Corollary 3.7. *For any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that for any weak solution u of (3.1) in Q_4 with*

$$\frac{1}{|Q_4|}\int_{Q_4} |\nabla u|^2\, dx\, dt \leqslant 1, \quad \frac{1}{|Q_4|}\int_{Q_4} \left(|\mathbf{f}|^2 + \left|A - \overline{A}_{Q_4}\right|^2\right) dx\, dt \leqslant \delta^2, \tag{3.16}$$

there exists a weak solution v of

$$v_t - div\left(\overline{A}_{Q_4}\nabla v\right) = 0 \tag{3.17}$$

in Q_4 such that

$$\|\nabla(u - v)\|^2_{L^2(Q_2)} \leqslant \varepsilon^2. \tag{3.18}$$

Proof. From (3.16) and Lemma 3.6, we see that for any $\eta > 0$, there is a small $\delta = \delta(\eta)$ and a corresponding weak solution v of

$$v_t - div\left(\overline{A}_{Q_4}\nabla v\right) = 0 \tag{3.19}$$

in Q_4 such that

$$\int_{Q_4} |(u - \overline{u}_{Q_4}) - v|^2 \leqslant \eta^2 \tag{3.20}$$

provided

$$\frac{1}{|Q_4|} \int_{Q_4} (|\mathbf{f}|^2 + |A - \overline{A}_{Q_4}|^2 \, dx \, dt \leqslant \delta^2.$$

Now we observe that $w = (u - \overline{u}_{Q_4}) - v$ is a weak solution of

$$w_t - div(A \nabla w) = div\left[\mathbf{f} - \left(A - \overline{A}_{Q_4}\right) \nabla v\right] \tag{3.21}$$

in Q_4. Then since $v \in W^{1,1}_\infty$, we see using (3.21) and Lemma 3.4 that

$$\begin{aligned}
\|\nabla(u - v)\|^2_{L^2(Q_2)} &\leqslant C(\|(u - \overline{u}_{Q_4}) - v\|^2_{L^2(Q_3)} + \left\|\mathbf{f} - \left(A - \overline{A}_{Q_4}\right) \nabla v\right\|^2_{L^2(Q_3)}) \\
&\leqslant C(\|(u - \overline{u}_{Q_4}) - v\|^2_{L^2(Q_3)} + \|\mathbf{f}\|^2_{L^2(Q_3)} + \left\|A - \overline{A}_{Q_4}\right\|^2_{L^2(Q_3)}) \\
&\leqslant C(\|(u - \overline{u}_{Q_4}) - v\|^2_{L^2(Q_4)} + \|\mathbf{f}\|^2_{L^2(Q_4)} + \left\|A - \overline{A}_{Q_4}\right\|^2_{L^2(Q_4)}).
\end{aligned}$$

Consequently this estimate and (3.20) imply

$$\begin{aligned}
\|\nabla(u - v)\|^2_{L^2(Q_2)} &\leqslant C\left(\eta^2 + |Q_4|\delta^2\right) \\
&= \varepsilon^2,
\end{aligned}$$

by taking η and δ satisfying the last identity above. This completes our proof. $\square$

We study local estimates of weak solutions

$$u_t - div(A \nabla u) = div\, \mathbf{f}$$

in Q_R by comparison with weak solutions of

$$v_t - div\left(\overline{A}_{Q_R} \nabla v\right) = 0$$

in Q_R and a real-variable argument based on the Vitali covering lemma.

Lemma 3.8. *There is a constant N_1 so that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if u is a weak solution of (3.1) in $\Omega_T \supset Q_7(0, 2)$ with*

$$Q_1 \cap \left\{(x, t) : \mathcal{M}(|\nabla u|^2) \leqslant 1\right\} \cap \left\{(x, t) : \mathcal{M}(|\mathbf{f}|^2) \leqslant \delta^2\right\} \neq \emptyset \tag{3.22}$$

and A uniformly parabolic and $(\delta, 7)$-vanishing, then

$$\left|\left\{(x, t) : \mathcal{M}(|\nabla u|^2) > N_1^2\right\} \cap Q_1\right| \leqslant \varepsilon |Q_1|. \tag{3.23}$$

Proof. From the condition (3.22), we see that there is a point $(x_0, t_0) \in Q_1$ such that

$$\frac{1}{|C_r|} \int_{C_r(x_0,t_0)\cap\Omega_T} |\nabla u|^2 \leqslant 1, \quad \frac{1}{|C_r|} \int_{C_r(x_0,t_0)\cap\Omega_T} |\mathbf{f}|^2 \leqslant \delta^2 \tag{3.24}$$

for all $r > 0$. Since $Q_4(0, 2) \subset C_6(x_0, t_0)$, we see from (3.24) that

$$\frac{1}{|Q_4|} \int_{Q_4(0,2)} |\mathbf{f}|^2 \, dx\, dt \leqslant \frac{|C_6|}{|Q_4|} \frac{1}{|C_6|} \int_{C_6(x_0,t_0)} |\mathbf{f}|^2 \, dx\, dt \leqslant (6/4)^{n+2} \delta^2. \tag{3.25}$$

Similarly, it follows that

$$\frac{1}{|Q_4|} \int_{Q_4(0,2)} |\nabla u|^2 \, dx\, dt \leqslant (6/4)^{n+2}. \tag{3.26}$$

In view of (3.25), (3.26) and from the assumption on A, we can apply Corollary 3.7 with u replaced by $(4/6)^{n+2} u$, $\mathbf{f}$ by $(4/6)^{n+2}\mathbf{f}$ and Q_4 by $Q_4(0, 2)$, respectively, to find that for any $\eta > 0$, there is a small $\delta = \delta(\eta)$ and a corresponding weak solution v of

$$v_t - div\left(\overline{A}_{Q_4(0,2)} \nabla v\right) = 0 \tag{3.27}$$

in $Q_4(0, 2)$ such that

$$\int_{Q_2(0,2)} |\nabla(u - v)|^2 \, dx\, dt \leqslant \eta^2 \tag{3.28}$$

provided

$$\frac{1}{|Q_4|} \int_{Q_4(0,2)} \left(|\mathbf{f}|^2 + \left|A - \overline{A}_{Q_4(0,2)}\right|^2\right) dx\, dt.$$

Now since $v \in W^{1,1}_{\infty}$, we may choose an appropriate constant N_0^2 such that

$$\sup_{(x,t)\in Q_3(0,2)} \{|\nabla v|^2(x,t)\} = N_0^2. \tag{3.29}$$

Setting $N_1^2 = \max\left\{4N_0^2, 2^{n+2}\right\}$, we claim that

$$\{(x,t) : \mathcal{M}(|\nabla u|^2) > N_1^2\} \cap Q_1 \subset \{(x,t) : \mathcal{M}_{Q_4(0,2)}(|\nabla(u-v)|^2) > N_0^2\} \cap Q_1. \tag{3.30}$$

To check this, suppose that

$$(x_1, t_1) \in \left\{(x,t) : \mathcal{M}_{Q_4(0,2)}(|\nabla(u-v)|^2) \leqslant N_0^2\right\} \cap Q_1. \tag{3.31}$$

For $r \leqslant 2$, $C_r(x_1,t_1) \subset Q_3(0,2)$, and so by (3.29) and (3.31), it follows that

$$\frac{1}{|C_r|}\int_{C_r(x_1,t_1)} |\nabla u|^2\,dx\,dt \leqslant \frac{2}{|C_r|}\int_{Q_3(0,2)} \left(|\nabla(u-v)|^2 + |\nabla v|^2\right)\,dx\,dt \leqslant 4N_0^2.$$

For $r > 2$, $C_r(x_1,t_1) \subset C_{2r}(x_0,t_0)$, and so by (3.24), it follows that

$$\frac{1}{|C_r|}\int_{C_r(x_1,t_1)\cap\Omega_T} |\nabla u|^2\,dx\,dt \leqslant \frac{1}{|C_r|}\int_{C_{2r}(x_0,t_0)\cap\Omega_T} |\nabla u|^2\,dx\,dt \leqslant 2^{n+2}.$$

This says that

$$(x_1,t_1) \in \left\{(x,t) : \mathcal{M}(|\nabla u|^2) \leqslant N_1^2\right\} \cap Q_1. \tag{3.32}$$

Then the assertion (3.30) follows from (3.31) and (3.32).

Finally, we calculate from (3.30) and parabolic weak $1-1$ estimates that

$$\begin{aligned}
|\{(x,t) : \mathcal{M}(|\nabla u|^2) > N_1^2\} \cap Q_1| &\leqslant |\{(x,t) : \mathcal{M}_{Q_4(0,2)}(|\nabla(u-v)|^2) > N_0^2\} \cap Q_1| \\
&\leqslant \frac{C}{N_0^2}\int_{Q_2(0,2)} |\nabla(u-v)|^2\,dx\,dt \\
&\leqslant \frac{C}{N_0^2}\eta^2 \\
&= \varepsilon|Q_1|,
\end{aligned}$$

by taking η and δ satisfying the last identity above. This completes our proof. $\square$

From Lemma 3.8 and the scaling argument, we obtain the following theorem.

Theorem 3.9. *Assume that u is a weak solution of (3.1) in Ω_T and let C_r be a centered parabolic cube centered at a point in Ω_T with $7C_r \subset \Omega_T$. Suppose further that*

$$\left|\left\{(x,t) : \mathcal{M}(|\nabla u|^2)(x,t) > N_1^2\right\} \cap C_r\right| \geqslant \varepsilon\,|C_r|\,.$$

Then we have

$$C_r \subset \left\{(x,t) : \mathcal{M}(|\nabla u|^2)(x,t) > 1\right\} \cup \left\{(x,t) : \mathcal{M}(|\mathbf{f}|^2)(x,t) > \delta^2\right\}.$$

Now we take N_1, ε, and the corresponding δ given by Theorem 3.9.

Corollary 3.10. *Suppose that u is a weak solution of (3.1) in $\Omega_T \supset Q_9(0,2)$ with the condition that*

$$\left|\left\{(x,t) \in \Omega_T : \mathcal{M}(|\nabla u|^2)(x,t) > N_1^2\right\}\right| < \varepsilon |Q_1|.$$

Let k be a positive integer and set $\varepsilon_1 = (10)^{n+2}\varepsilon$. Then we have

$$\begin{aligned} |\{(x,t) \in Q_1 : \mathcal{M}(|\nabla u|^2) > N_1^{2k}\}| \leqslant\ & \sum_{i=1}^{k} \varepsilon_1^i |\{(x,t) \in Q_1 : \mathcal{M}(|\mathbf{f}|^2) > \delta^2 N_1^{2(k-i)}\}| \\ & + \varepsilon_1^k |\{(x,t) \in Q_1 : \mathcal{M}(|\nabla u|^2) > 1\}|. \end{aligned}$$

Proof. We will prove this corollary by induction on k. The case $k = 1$ comes from Theorem 3.9 and Lemma 2.5 with

$$\begin{aligned} E &= \{(x,t) \in Q_1 : \mathcal{M}(|\nabla u|^2)(x,t) > N_1^2\}, \\ F &= \{(x,t) \in Q_1 : \mathcal{M}(|\mathbf{f}|^2)(x,t) > \delta^2\} \cup \{(x,t) \in Q_1 : \mathcal{M}(|\nabla u|^2)(x,t) > 1\}. \end{aligned}$$

Assume now that the conclusion is valid for some positive integer k. Let us define $u_1 = u/N_1$ and corresponding $\mathbf{f}_1 = \mathbf{f}/N_1$. Then u_1 is a weak solution of (3.1) in $Q_7(0,2)$ and satisfies

$$\left|\left\{(x,t) \in \Omega_T \ : \mathcal{M}(|\nabla u_1|^2)(x,t) > N_1^2\right\}\right| < \varepsilon |Q_1|.$$

Then by induction hypothesis, we have the following estimates:

$$
\begin{aligned}
&|\{(x,t) \in Q_1 : \mathcal{M}(|\nabla u|^2) > N_1^{2(k+1)}\}| \\
&\quad = |\{(x,t) \in Q_1 : \mathcal{M}(|\nabla u_1|^2) > N_1^{2k}\}| \\
&\quad \leqslant \sum_{i=1}^{k} \varepsilon_1^i |\{(x,t) \in Q_1 : \mathcal{M}(|\mathbf{f}_1|^2) > \delta^2 N_1^{2(k-i)}\}| \\
&\qquad + \varepsilon_1^k |\{(x,t) \in Q_1 : \mathcal{M}(|\nabla u_1|^2) > 1\}| \\
&\quad = \sum_{i=1}^{k+1} \varepsilon_1^i |\{(x,t) \in Q_1 : \mathcal{M}(|\mathbf{f}|^2) > \delta^2 N_1^{2(k+1-i)}\}| \\
&\qquad + \varepsilon_1^{k+1} |\{(x,t) \in Q_1 : \mathcal{M}(|\nabla u|^2) > 1\}|
\end{aligned}
$$

and so the conclusion is valid for $k+1$, which completes our induction argument. $\square$

We are now ready to prove Theorem 3.2. The primary technical tools are standard arguments of measure theory (see [10]) and Corollary 3.10.

Proof of Theorem 3.2. Without loss of generality, we may assume that

$$
\left| \left\{ (x,t) : \mathcal{M}(|\nabla u|^2)(x,t) > N_1^2 \right\} \cap Q_1 \right| < \varepsilon |Q_1| \tag{3.33}
$$

and

$$
\|\mathbf{f}\|_{L^p(Q_7(0,2))} \text{ is small enough,} \tag{3.34}
$$

by multiplying the PDE (3.1) by a small constant depending on $\|\mathbf{f}\|_{L^p(Q_7(0,2))}$ and $\|\nabla u\|_{L^p(Q_7(0,2))}$. Since $\mathbf{f} \in L^p(Q_7(0,2))$, it follows from strong $p-p$ estimates that

$$
\mathcal{M}(|\mathbf{f}|^2) \in L^{\frac{p}{2}}(Q_7(0,2)).
$$

Consequently from (3.34), we calculate that

$$
\sum_{k=0}^{\infty} N_1^{pk} \left| \left\{ (x,t) : \mathcal{M}(|\mathbf{f}|^2)(x,t) > \delta^2 N_1^{2k} \right\} \right| \leqslant C \|\mathcal{M}(|\mathbf{f}|^2)\|_{L^{\frac{p}{2}}(Q_9(0,2))}^{\frac{p}{2}} \leqslant 1. \tag{3.35}
$$

Then from Corollary 3.10 and (3.35) we calculate

$$
\begin{aligned}
&\sum_{k=1}^{\infty} N_1^{pk} |\{(x,t) \in Q_1 : \mathcal{M}(|\nabla u|^2) > N_1^{2k}\}| \\
&\quad \leqslant \sum_{k=1}^{\infty} N_1^{pk} \left(\sum_{i=1}^{k} \varepsilon_1^i \left| \left\{ x \in Q_1 : \mathcal{M}(|\mathbf{f}|^2)(x,t) > \delta^2 N_1^{2(k-i)} \right\} \right| \right. \\
&\qquad \left. + \varepsilon_1^k \left| \left\{ x \in Q_1 : \mathcal{M}(|\nabla u|^2)(x,t) > 1 \right\} \right| \right) \\
&\quad = \sum_{i=1}^{\infty} (N_1^p \varepsilon_1)^i \left(\sum_{k=i}^{\infty} N_1^{p(k-i)} |\{(x,t) \in Q_1 : \mathcal{M}(|\nabla u|^2) > \delta^2 N_1^{2(k-i)}\}| \right) \\
&\qquad + \sum_{k=1}^{\infty} (N_1^p \varepsilon_1)^k |\{(x,t) \in Q_1 : \mathcal{M}(|\nabla u|^2) > 1\}| \\
&\quad \leqslant C \sum_{k=1}^{\infty} (N_1^p \varepsilon_1)^k;
\end{aligned}
$$

that is,

$$
\sum_{k=1}^{\infty} N_1^{pk} |\{(x,t) \in Q_1 : \mathcal{M}(|\nabla u|^2) > N_1^{2k}\}| \leqslant C \sum_{k=1}^{\infty} (N_1^p \varepsilon_1)^k.
$$

Now select ε_1 so that $N_1^p \varepsilon_1 < 1$ to obtain

$$
\mathcal{M}(|\nabla u|^2) \in L^{p/2}(Q_7(0,2))
$$

by standard arguments of measure theory (see [10, Lemma 7.3]), and so a fortiori $\nabla u \in L^p(Q_7(0,2))$.

Remark 3.11. It is possible to select ε_1 so that $N_1^p \varepsilon_1 < 1$ since N_1 is a universal constant depending on the dimension and parabolicity, and since p is given. So we can take an appropriate ε and ε_1.

4. The Dirichlet problem in Lipschitz domains

In this section, we extend the interior estimates established in the previous section to study the smoothness of weak solutions up to the boundary. The assumptions considered in this work are that the matrix of coefficients is (δ, R)-vanishing; that is, the coefficients of the operator have small BMO semi-norms, and that the domain is (δ, R)-Lipschitz; that is, $\partial\Omega$ is locally given by graphs with small Lipschitz constants. We first investigate the special case that Ω is a half-ball, $\Omega = B_R^+(0)$, with $R > 1$, to obtain the boundary L^p estimates for the gradient of our weak solution u in $Q_1^+ = B_1^+ \times (-1, 0]$. Then in the next section by standard scaling, covering and flattening arguments, we obtain the estimates on the lateral boundary. For the estimates on the bottom and corner of the boundary, we just extend the solution by 0. Taking Theorem 2.6 into account we will use the interplay between the analytic properties of the coefficients and the geometric properties of the domain.

For our purpose we localize our interest on Q_R^+ and on a weak solution of

$$\begin{cases} u_t - div(A\nabla u) = div\,\mathbf{f} & \text{in} \quad Q_R^+, \\ u = 0 & \text{on} \quad \hat{T}_R, \end{cases} \tag{4.1}$$

and a weak solution of the corresponding approximation PDE

$$\begin{cases} v_t - div\left(\overline{A}_{Q_R^+}\nabla v\right) = 0 & \text{in} \quad Q_R^+, \\ v = 0 & \text{on} \quad \hat{T}_R. \end{cases} \tag{4.2}$$

We start with the following classical theory concerning divergence form parabolic equations.

Definition 4.1. We say that $u \in \mathring{W}_*^{1,2}(\Omega_T)$ is a weak solution of (1.1) if

$$\int_{\Omega_T} u\varphi_t\,dx\,dt - \int_{\Omega_T} A\nabla u\nabla\varphi\,dx\,dt = \int_{\Omega_T} \mathbf{f}\nabla\varphi\,dx\,dt$$

for all $\varphi \in C_0^\infty(\Omega_T)$ with $\varphi = 0$ on the top of Ω_T.

Theorem 4.2 (*Baiocchi [5], Ladyzhenskaya [19] and Lieberman [20]*). *There exists a unique weak solution of* (1.1).

Definition 4.3. (1) We say that $u \in W_*^{1,2}(Q_R^+)$ is a weak solution of (4.1) if

$$\int_{Q_R^+} u\varphi_t\,dx\,dt - \int_{Q_R^+} A\nabla u\nabla\varphi\,dx\,dt = \int_{Q_R^+} \mathbf{f}\nabla\varphi\,dx\,dt$$

for all $\varphi \in C_0^\infty(Q_R^+)$ with $\varphi = 0$ for $t = 0$ and the zero extension of u is in $W_*^{1,2}(Q_R)$.

(2) We say that $v \in W^{1,2}_*(Q^+_R)$ is a weak solution of (4.2) if

$$\int_{Q^+_R} v\varphi_t \, dx\, dt - \int_{Q^+_R} \overline{A}_{Q^+_R} \nabla v \nabla \varphi \, dx\, dt = 0$$

for all $\varphi \in C^\infty_0(Q^+_R)$ with $\varphi = 0$ for $t = 0$ and the zero extension of v is in $W^{1,2}_*(Q_R)$.

Remark 4.4. We remark that our definition of a weak solution u is actually equivalent to the popular one in [16]; that is, we say a function

$$u \in L^2(-R^2, 0; H^1_0(B^+_R)) \quad \text{with} \quad u_t \in L^2(-R^2, 0; H^{-1}(B^+_R))$$

is a weak solution of (4.1) if

$$\langle u_t, \varphi\rangle + \int_{B^+_R} A\nabla u \nabla\varphi \, dx = -\int_{B^+_R} \mathbf{f}\nabla\varphi \, dx$$

for each $\varphi \in H^1_0(B^+_R)$ and a.e. time $-R^2 \leqslant t \leqslant 0$, where $\langle , \rangle$ is the paring of $H^{-1}(B^+_R)$ and $H^1_0(B^+_R)$.

Remark 4.5. We remark that a solution of (4.2) satisfies an interior $W^{1,1}_\infty$ regularity; that is, ∇u and u_t are all uniformly(essentially) bounded.

The following energy estimate is used later in Corollary 4.10.

Lemma 4.6. *Assume that $u \in W^{1,2}_*(Q^+_2)$ is a weak solution of* (4.1). *Then we have*

$$\int_{Q^+_1} |\nabla u|^2 \, dx\, dt \leqslant C \left(\int_{Q^+_2} (|\mathbf{f}|^2 + |u|^2) \, dx\, dt \right).$$

Proof. Temporarily suppose that u is a smooth function and let $\eta = \eta(x,t)$ be a smooth cut-off function; that is,

$$0 \leqslant \eta \leqslant 1, \;\; \eta = 1 \text{ on } \overline{Q_1} \quad \text{and} \quad \eta = 0 \text{ near } \partial_p Q_2. \tag{4.3}$$

Then one can replace the test function φ by $\phi^2 u$ in Remark 4.4; that is, one can multiply Eq. (4.1) by $\eta^2 u$. Then we see from the integration by parts formula over B^+_2 that

$$\int_{B^+_2} u_t(\eta^2 u)\, dx + \int_{B^+_2} A\nabla u \nabla(\eta^2 u)\, dx = -\int_{B^+_2} \mathbf{f}\nabla(\eta^2 u)\, dx.$$

We write the resulting expression as

$$I_1 + I_2 = I_3 + I_4$$

for

$$\begin{aligned}
I_1 &= \frac{d}{dt}\int_{B_2^+} \eta^2 \frac{|u|^2}{2}\, dx,\\
I_2 &= \int_{B_2^+} \eta^2 (A\nabla u \nabla u)\, dx,\\
I_3 &= \int_{B_2^+} \eta\eta_t |u|^2\, dx - 2\int_{B_2^+} \eta u (A\nabla u\nabla \eta)\, dx,\\
I_4 &= -\int_{B_2^+} \mathbf{f}\nabla(\eta^2 u)\, dx.
\end{aligned}$$

Estimate of I_2: From the uniform parabolicity condition (see Definition 2.1), we see that

$$\begin{aligned}
I_2 &= \int_{B_2^+} \eta^2 \ (A\nabla u\nabla u)\, dx\\
&\geqslant \Lambda^{-1}\int_{B_2^+} \eta^2 |\nabla u|^2\, dx.
\end{aligned}$$

Estimate of I_3: As $A \in L^\infty$, we see from (4.3) and Cauchy's inequality with τ that

$$\begin{aligned}
I_3 &= \int_{B_2^+} \eta\eta_t |u|^2\, dx - 2\int_{B_2^+} \eta u (A\nabla u\nabla\eta)\, dx\\
&\leqslant C\left(1+\frac{1}{\tau}\right)\int_{B_2^+} |u|^2\, dx + C\ \tau\int_{B_2^+} \eta^2|\nabla u|^2\, dx.
\end{aligned}$$

Estimate of I_4: Cauchy's inequality with τ implies that

$$\begin{aligned}
I_4 &= -\int_{B_2^+} \mathbf{f}\nabla(\eta^2 u)\, dx\\
&= -\int_{B_2^+} \Big((\mathbf{f}\nabla u)\eta^2 + 2(\mathbf{f}\nabla\eta)\eta u\Big)\ dx\\
&\leqslant \tau\int_{B_2^+} \eta^2|\nabla u|^2\, dx + \frac{1}{4\tau}\int_{B_2^+} |\mathbf{f}|^2\, dx + C\int_{B_2^+} \Big(|u|^2 + |\mathbf{f}|^2\Big)\ dx\\
&\leqslant \tau\int_{B_2^+} \eta^2|\nabla u|^2\, dx + C\int_{B_2^+} |u|^2\, dx + C\left(1+\frac{1}{\tau}\right)\int_{B_2^+} |\mathbf{f}|^2\, dx.
\end{aligned}$$

We then combine of the estimates I_i $(1 \leqslant i \leqslant 4)$ to discover

$$\begin{aligned}
&\frac{d}{dt}\int_{B_2^+} \eta^2 \frac{|u|^2}{2}\, dx + \Lambda^{-1}\int_{B_2^+} \eta^2 |\nabla u|^2\, dx \\
&\quad \leqslant I_1 + I_2 = I_3 + I_4 \\
&\quad \leqslant C\left(1+\frac{1}{\tau}\right)\int_{B_2^+} |u|^2\, dx + C\ \tau \int_{B_2^+} \eta^2 |\nabla u|^2\, dx \\
&\qquad + \tau \int_{B_1^+} \eta^2 |\nabla u|^2\, dx + C\int_{B_1^+} |u|^2\, dx + C\left(1+\frac{1}{\tau}\right)\int_{B_2^+} |\mathbf{f}|^2\, dx \\
&\quad \leqslant C\left(1+\frac{1}{\tau}\right)\int_{B_2^+} \left(|u|^2 + |\mathbf{f}|^2\right) dx + C\ \tau \int_{B_2^+} \eta^2 |\nabla u|^2\, dx.
\end{aligned}$$

We take τ small enough to see that

$$\frac{d}{dt}\int_{B_2^+} \eta^2 \frac{|u|^2}{2}\, dx + \int_{B_2^+} \eta^2 |\nabla u|^2\, dx \leqslant C \int_{B_2^+} \left(|u|^2 + |\mathbf{f}|^2\right) dx.$$

Integrating with respect to time from -4 to 0 and noting (4.3), we finally obtain

$$\int_{Q_1^+} |\nabla u|^2\, dx\, dt \leqslant C \int_{Q_2^+} \left(|u|^2 + |\mathbf{f}|^2\right) dx\, dt.$$

By approximation we find the same estimates hold with the smooth function u replaced by our weak solution; that is, the test function involves the Steklov average of u:

$$u_h(x,t) = \frac{1}{h}\int_0^h u(x, t+\tau)\, d\tau,$$

and this expression leads to a suitable test function since the domain considered in this paper is cylindrical. For further discussion concerning this issue, we refer to the Chapter 3 in [19] and the Chapter 6 in [20]. □

Theorem 4.7. *Let $u \in W_*^{1,2}(Q_1^+)$ be a weak solution of the parabolic PDE* (4.1). *Then we have*

$$\|u\|^2_{L^2(Q_1^+)} \leqslant C\left(\|\nabla u\|^2_{L^2(Q_1^+)} + \|\mathbf{f}\|^2_{L^2(Q_1^+)}\right).$$

Proof. We prove it by contradiction. If not, there exist $\{A_k\}_{k=1}^{\infty}$, $\{u_k\}_{k=1}^{\infty}$ and $\{\mathbf{f}_k\}_{k=1}^{\infty}$ such that $u_k \in W_*^{1,2}(Q_1^+)$ is a weak solution of

$$\begin{cases} (u_k)_t - div(A_k \nabla u_k) = div\, \mathbf{f}_k & \text{in} \quad Q_1^+, \\ u_k = 0 & \text{on} \quad \hat{T}_1 \end{cases} \tag{4.4}$$

with

$$\|u_k\|^2_{L^2(Q_1^+)} \geqslant k \left(\|\nabla u_k\|^2_{L^2(Q_1^+)} + \|\mathbf{f}_k\|^2_{L^2(Q_1^+)} \right).$$

Normalize u_k so that $\|u_k\|_{L^2(Q_1^+)} = 1$ to obtain

$$\|u_k\|^2_{W_*^{1,2}(Q_1^+)} \leqslant C(\|u_k\|^2_{L^2(Q_1^+)} + \|\nabla u_k\|^2_{L^2(Q_1^+)} + \|\mathbf{f}_k\|^2_{L^2(Q_1^+)}) \leqslant C(1 + 1/k) \leqslant C$$

and so deduce

$$\|\nabla u_k\|^2_{L^2(Q_1^+)} + \|\mathbf{f}_k\|^2_{L^2(Q_1^+)} \leqslant \frac{1}{k} \to 0 \quad \text{as} \quad k \to \infty. \tag{4.5}$$

Let u_0 be the weak limit of $\{u_k\}$. Then we have, up to a subsequence,

$$\begin{cases} u_k \to u_0 \text{ with } \|u_0\|_{L^2(Q_1^+)} = 1 & \text{in} \quad L^2(Q_1^+), \\ \nabla u_k \rightharpoonup \nabla u_0 (= 0) & \text{in} \quad L^2(Q_1^+), \\ u_k \rightharpoonup u_0 & \text{in} \quad W_*^{1,2}(Q_1^+). \end{cases} \tag{4.6}$$

We claim that u_0 is a weak solution of

$$\begin{cases} (u_0)_t = 0 & \text{in} \quad Q_1^+, \\ u_0 = 0 & \text{on} \quad \hat{T}_1. \end{cases} \tag{4.7}$$

To show this, choose any $\varphi \in C_0^{\infty}(Q_1^+)$ with $\varphi = 0$ for $t = 0$. Then by (4.4), we obtain

$$\int_{Q_1^+} u_k \varphi_t \, dx\, dt - \int_{Q_1^+} A_k \nabla u_k \nabla \varphi \, dx\, dt = \int_{Q_1^+} \mathbf{f}_k \nabla \varphi \, dx\, dt. \tag{4.8}$$

Let $k \to \infty$ in (4.8) to find

$$\int_{Q_1^+} u_0 \varphi_t \, dx\, dt = 0,$$

which shows (4.7). Then in light of (4.7) and (4.6), $u_0 = 0$, and so we reach a contradiction to (4.6). $\square$

Lemma 4.8. *Let $u \in W_*^{1,2}(Q_2^+)$ be a weak solution of the parabolic PDE (4.1). Then we have*

$$\|u\|^2_{W_*^{1,2}(Q_1^+)} \leqslant C\left(\|\nabla u\|^2_{L^2(Q_1^+)} + \|\mathbf{f}\|^2_{L^2(Q_1^+)}\right).$$

Proof. We turn to the PDE (4.1) to invoke Definition 2.4. Then we apply Theorem 4.7 to obtain the following estimates:

$$\begin{aligned}\|u\|^2_{W_*^{1,2}(Q_1^+)} &\leqslant \|u\|^2_{L^2(Q_1^+)} + \|\nabla u\|^2_{L^2(Q_1^+)} + \|(A\nabla u + \mathbf{f})\|^2_{L^2(Q_1^+)} \\ &\leqslant C\left(\|\nabla u\|^2_{L^2(Q_2^+)} + \|\mathbf{f}\|^2_{L^2(Q_2^+)}\right). \qquad \square\end{aligned}$$

Lemma 4.9. *For any $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ such that for any weak solution $u \in W_*^{1,2}(Q_4^+)$ of (4.1) with*

$$\frac{1}{|Q_4|}\int_{Q_4^+} |\nabla u|^2\, dx\, dt \leqslant 1, \quad \frac{1}{|Q_4|}\int_{Q_4^+}\left(|\mathbf{f}|^2 + \left|A - \overline{A}_{Q_4^+}\right|^2\right) dx\, dt \leqslant \delta^2,$$

there exists a weak solution v of (4.2) in Q_4^+ such that

$$\int_{Q_4^+} |u - v|^2\, dx\, dt \leqslant \varepsilon^2.$$

Proof. If not, there exist $\varepsilon_0 > 0$, $\{A_k\}_{k=1}^\infty$, $\{u_k\}_{k=1}^\infty$, and $\{\mathbf{f}_k\}_{k=1}^\infty$ such that u_k is a weak solution of

$$\begin{cases} (u_k)_t - div(A_k \nabla u_k) = div\, \mathbf{f}_k & \text{in} \quad Q_4^+, \\ u_k = 0 & \text{on} \quad \hat{T}_4, \end{cases} \tag{4.9}$$

with

$$\frac{1}{|Q_4|}\int_{Q_4^+} |\nabla u_k|^2\, dx\, dt \leqslant 1, \quad \frac{1}{|Q_4|}\int_{Q_4^+}\left(|\mathbf{f}_k|^2 + \left|A_k - \overline{A_k}_{Q_4^+}\right|^2\right) dx\, dt \leqslant \frac{1}{k^2}. \tag{4.10}$$

But,

$$\int_{Q_4^+} |u_k - v|^2 \, dx \, dt > \varepsilon_0^2 \tag{4.11}$$

for any weak solution v of (4.2) in Q_4^+.

According to (4.9), Lemma 4.8 and (4.10), $\{u_k\}_{k=1}^{\infty}$ is bounded in $W_*^{1,2}\left(Q_4^+\right)$. Thus there exists a subsequence, which we denote by $\{u_k\}$, such that

$$u_k \rightharpoonup u_0 \text{ in } W_*^{1,2}(Q_4^+) \quad \text{ and } \quad u_k \to u_0 \text{ in } L^2(Q_4^+) \tag{4.12}$$

for some u_0 in $W_*^{1,2}(Q_4^+)$ with $u_0 = 0$ on $\hat{T}_4$. Since $\left\{\overline{A_k}_{Q_4^+}\right\}_{k=1}^{\infty}$ is bounded in ℓ^∞, it has a subsequence, which we denote by $\left\{\overline{A_k}\right\}$, such that

$$\left\|\overline{A_k} - A_0\right\|_{\infty} \to 0 \quad \text{as} \quad k \to \infty. \tag{4.13}$$

But then, by (4.13) and (4.10), we have

$$A_k \;\to\; A_0 \quad \text{in} \quad L^2\left(Q_4^+\right). \tag{4.14}$$

Now we will show that u_0 is a weak solution of

$$\begin{cases} (u_0)_t - div(A_0 \nabla u_0) = 0 & \text{in} \quad Q_4^+, \\ u_0 = 0 & \text{on} \quad \hat{T}_4. \end{cases} \tag{4.15}$$

To do this, fix any φ in $C_0^\infty(Q_4^+)$ with $\varphi = 0$ for $t = 0$. Then, by (4.9), we have

$$\int_{Q_4^+} u_k \varphi_t \, dx \, dt - \int_{Q_4^+} A_k \nabla u_k \nabla \varphi \, dx \, dt = \int_{Q_4^+} \mathbf{f}_k \nabla \varphi \, dx \, dt. \tag{4.16}$$

Now let $k \to \infty$ to find

$$\int_{Q_4^+} u_0 \varphi_t \, dx \, dt - \int_{Q_4^+} A_0 \nabla u_0 \nabla \varphi \, dx \, dt = 0, \tag{4.17}$$

in view of (4.12) and (4.10). This is (4.15).

Finally, we have a contradiction to (4.11) by taking $v = u_0$ and k large enough as we did in the proof of Lemma 3.6. $\square$

Corollary 4.10. *For any* $\varepsilon > 0$, *there exists a small* $\delta = \delta(\varepsilon) > 0$ *such that for any weak solution* $u \in W_*^{1,2}(Q_4^+)$ *of* (4.1) *with*

$$\frac{1}{|Q_4|}\int_{Q_4^+} |\nabla u|^2\,dx\,dt \leqslant 1, \quad \frac{1}{|Q_4|}\int_{Q_4^+}\left(|\mathbf{f}|^2 + \left|A - \overline{A}_{Q_4^+}\right|^2\right)dx\,dt \leqslant \delta^2, \tag{4.18}$$

there exist a weak solution v *of* (4.2) *in* Q_4^+ *such that*

$$\int_{Q_4^+} |\nabla(u-v)|^2\,dx\,dt \leqslant \varepsilon^2.$$

Proof. By Lemma 4.9 and (4.18), for any $\eta > 0$, there exists $\delta = \delta(\eta)$, and a corresponding weak solution v of (4.2) in Q_4^+ such that

$$\int_{Q_4^+} |u-v|^2\,dx\,dt \leqslant \eta^2 \tag{4.19}$$

provided

$$\frac{1}{|Q_4|}\int_{Q_4^+}\left(|\mathbf{f}|^2 + \left|A - \overline{A}_{Q_4^+}\right|^2\right)dx\,dt \leqslant \delta^2.$$

First we observe that $u - v$ is a weak solution of

$$(u-v)_t - div(A\nabla(u-v)) = div\left[\mathbf{f} - \left(A - \overline{A}_{Q_4^+}\right)\nabla v\right] \tag{4.20}$$

in Q_4^+ with $u - v = 0$ on $\hat{T}_4$. Then by Lemma 4.6, we have

$$\begin{aligned}
\|\nabla(u-v)\|^2_{L^2(Q_2^+)} &\leqslant C\left(\|u-v\|^2_{L^2(Q_3^+)} + \left\|\mathbf{f} - \left(A - \overline{A}_{Q_4^+}\right)\nabla v\right\|^2_{L^2(Q_3^+)}\right)\\
&\leqslant C\left(\|u-v\|^2_{L^2(Q_3^+)} + \|\mathbf{f}\|^2_{L^2(Q_3^+)} + \left\|A - \overline{A}_{Q_4^+}\right\|^2_{L^2(Q_3^+)}\right)\\
&\leqslant C\left(\|u-v\|^2_{L^2(Q_4^+)} + \|\mathbf{f}\|^2_{L^2(Q_4^+)} + \left\|A - \overline{A}_{Q_4^+}\right\|^2_{L^2(Q_4^+)}\right).
\end{aligned}$$

This estimate and (4.19) imply

$$\begin{aligned}
\|\nabla(u-v)\|^2_{L^2(Q_2^+)} &\leqslant C\left(\eta^2 + |Q_4^+|\delta^2\right)\\
&\leqslant \varepsilon^2
\end{aligned}$$

by taking η and δ satisfying the last inequality above. This completes our proof. $\square$

Now we invoke our notation $S\Omega_T = \Omega \times [a, a+T]$ $(a > 0)$ for the lateral boundary of Ω_T. We refer to only the estimates on the lateral boundary since the zero extension of our solution can lead to the estimates on the bottom and corner of the boundary.

Lemma 4.11. *There is a constant $N_1 > 0$ so that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ with A uniformly parabolic and $(\delta, 7)$-vanishing, and if $u \in \mathring{W}^{1,2}_{*}(\Omega_T)$ is a weak solution of (1.1) with $\Omega_T \supset Q_7^+(0,2)$, $\Omega_T \supset \hat{T}_7(0,2)$ and*

$$Q_1^+ \cap \left\{(x,t) : \mathcal{M}\left(|\nabla u|^2\right)(x,t) \leqslant 1\right\} \cap \left\{(x,t) : \mathcal{M}\left(|\mathbf{f}|^2\right)(x,t) \leqslant \delta^2\right\} \neq \emptyset, \tag{4.21}$$

then

$$\left|\left\{(x,t) : \mathcal{M}\left(|\nabla u|^2\right)(x,t) > N_1^2\right\} \cap Q_1^+\right| \leqslant \varepsilon \left|Q_1^+\right|. \tag{4.22}$$

Proof. From (4.21), it follows that there is a point $(x_0, t_0) \in Q_1^+$ such that

$$\frac{1}{|C_r|}\int_{K_r^+(x_0,t_0)\cap\Omega_T} |\nabla u|^2\, dx\, dt \leqslant 1, \quad \frac{1}{|C_r|}\int_{C_r^+(x_0,t_0)\cap\Omega_T} |\mathbf{f}|^2\, dx\, dt \leqslant \delta^2 \tag{4.23}$$

for all $r > 0$. Since $Q_4^+(0,2) \subset C_6^+(x_0,t_0) \cap \Omega_T$, from (1.1), it follows that

$$\frac{1}{|Q_4|}\int_{Q_4^+(0,2)} |\mathbf{f}|^2\, dx\, dt \leqslant \frac{1}{|C_6|}\int_{C_6^+(x_0,t_0)\cap\Omega_T} |\mathbf{f}|^2\, dx\, dt \leqslant (6/4)^{n+2}\delta^2. \tag{4.24}$$

Similarly, it follows that

$$\frac{1}{|Q_4|}\int_{Q_4^+(0,2)} |\nabla u|^2\, dx\, dt \leqslant (6/4))^{n+2}. \tag{4.25}$$

Then in view of (4.24), (4.25) and our assumption that A is $(\delta, 7)$-vanishing, we may apply Corollary 4.10 when u is replaced by $(4/6)^{n+2}u$, $\mathbf{f}$ by $4/6)^{n+2}\mathbf{f}$, and Q_4^+ by $Q_4^+(0,2)$, respectively, to obtain that for any $\eta > 0$, there exist a small $\delta = \delta(\eta) > 0$ and a corresponding weak solution v of (4.2) in $Q_4^+(0,2)$ such that

$$\int_{Q_2^+(0,2)} |\nabla(u - v)|^2\, dx\, dt \leqslant \eta^2 \tag{4.26}$$

provided

$$\frac{1}{|Q_4|}\int_{Q_4^+(0,2)} \left(|\mathbf{f}|^2 + \left|A - \overline{A}_{Q_4^+(0,2)}\right|^2\right) dx\, dt \leqslant \delta^2.$$

Then by the standard local derivative estimates for v, we have

$$\sup_{(x,t)\in Q_3^+(0,2)} \{|\nabla v|^2(x,t)\} = N_0^2 \tag{4.27}$$

for some appropriate constant N_0. Write $N_1^2 = \max\left\{4N_0^2, 2^{n+2}\right\}$. We claim that

$$\{(x,t)\in Q_1^+ : \mathcal{M}(|\nabla u|^2) > N_1^2\} \subset \{(x,t) : \mathcal{M}_{Q_4^+(0,2)}(|\nabla(u-v)|^2) > N_0^2\}. \tag{4.28}$$

To prove this, suppose that

$$(x_1,t_1) \in \left\{(x,t)\in Q_1^+ : \mathcal{M}_{Q_4^+(0,2)}(|\nabla(u-v)|^2)(x,t) \leqslant N_0^2\right\}. \tag{4.29}$$

If $r\leqslant 2$, $C_r^+(x_1,t_1)\cap\Omega_T \subset Q_3^+(0,2)$ and so by (4.29) and (4.27), we have

$$\begin{aligned}\frac{1}{|C_r|}\int_{C_r^+(x_1,t_1)\cap\Omega_T} |\nabla u|^2\,dx\,dt \;&\leqslant\; \frac{2}{|C_r|}\int_{C_r^+(x_1,t_1)\cap\Omega_T} \left(|\nabla(u-v)|^2 + |\nabla v|^2\right)dx\,dt\\ &\leqslant\; 2N_0^2 + 2N_0^2 = 4N_0^2.\end{aligned}$$

If $r>2$, $C_r^+(x_1,t_1) \subset C_{2r}^+(x_0,t_0)$ and so by (4.23), we have

$$\frac{1}{|C_r|}\int_{C_r^+(x_1,t_1)\cap\Omega_T} |\nabla u|^2\,dx\,dt \leqslant \frac{1}{|C_r|}\int_{C_{2r}^+(x_0,t_0)\cap\Omega_T} |\nabla u|^2\,dx\,dt \leqslant 2^{n+2}.$$

Thus

$$(x_1,t_1) \in \left\{(x,t)\in Q_1^+ : \mathcal{M}(|\nabla u|^2)(x,t) \leqslant N_1^2\right\}. \tag{4.30}$$

The assertion (4.28) in turn comes from (4.29) and (4.30).

Finally, from (4.28) and parabolic weak $1-1$ estimate, we have the following estimates:

$$\begin{aligned}|\{(x,t)\in Q_1^+ : \mathcal{M}(|\nabla u|^2) > N_1^2\}| \;&\leqslant\; |\{(x,t)\in Q_1^+ : \mathcal{M}_{Q_4^+(0,2)}(|\nabla(u-v)|^2) > N_0^2\}|\\ &\leqslant\; \frac{C}{N_0^2}\int_{Q_2^+(0,2)} |\nabla(u-v)|^2\,dx\,dt\\ &\leqslant\; \frac{C}{N_0^2}\eta^2\\ &\leqslant\; \varepsilon\left|Q_1^+\right|,\end{aligned}$$

by taking η and δ satisfying the last inequality above. This finishes our proof. $\square$

We can now apply Lemma 4.11 from the perspective of the scaling argument to deduce the following lemma.

Lemma 4.12. *There is a constant $N_1 > 0$ so that for any $\varepsilon, r > 0$, there exists $\delta = \delta(\varepsilon) > 0$ with A uniformly parabolic and $(\delta, 7r)$-vanishing, and if $u \in \mathring{W}^{1,2}_*(\Omega_T)$ is a weak solution of (1.1) with $\Omega_T \supset Q^+_{7r}(0, 2r^2)$, $S\Omega_T \supset \hat{T}_{7r}(0, 2r^2)$ and*

$$Q_r^+ \cap \{(x,t) : \mathcal{M}(|\nabla u|^2) \leqslant 1\} \cap \{(x,t) : \mathcal{M}(|\mathbf{f}|^2) \leqslant \delta^2\} \neq \emptyset, \tag{4.31}$$

then

$$|\{(x,t) : \mathcal{M}(|\nabla u|^2) > N_1^2\} \cap Q_r^+| \leqslant \varepsilon |Q_r^+|. \tag{4.32}$$

Remark 4.13. We should point out that the Lemma 4.12 is only significant for small $r > 0$. In fact, we establish interior and boundary estimates for the gradient of u in small cubes.

Theorem 4.14. *There is a constant $N_1 > 0$ so that for any ε, r $(1 \geqslant \varepsilon, r > 0)$, there exists a small $\delta = \delta(\varepsilon) > 0$ with A uniformly parabolic and $(\delta, 1)$-vanishing, if $u \in \mathring{W}^{1,2}_*(\Omega_T)$ is a weak solution of (1.1) and if the following property*

$$\left| \left\{ (x,t) \in \Omega_T : \mathcal{M}(|\nabla u|^2)(x,t) > N_1^2 \right\} \cap C_r \right| \geqslant \varepsilon |C_r| \tag{4.33}$$

holds, then

$$C_r \cap \Omega_T \subset \{(x,t) : \mathcal{M}(|\nabla u|^2)(x,t) > 1\} \cup \{(x,t) : \mathcal{M}(|\mathbf{f}|^2)(x,t) > \delta^2\}, \tag{4.34}$$

where C_r is a cube centered at a point in Ω_T or on the lateral boundary $S\Omega_T$, of radius ρ and height ρ^2.

Proof. We prove this by contradiction. If C_r satisfies (4.33) and the conclusion (4.34) is false, then there exists $(x_0, t_0) \in C_r \cap \Omega_T$ such that

$$\frac{1}{|C_\rho|} \int_{C_\rho(x_0,t_0) \cap \Omega_T} |\nabla u|^2 \, dx\, dt \leqslant 1, \quad \frac{1}{|C_\rho|} \int_{C_\rho(x_0,t_0) \cap \Omega_T} |\mathbf{f}|^2 \, dx\, dt \leqslant \delta^2$$

for all $\rho > 0$. If $7C \cap \partial_p \Omega_T = \emptyset$, this is an interior estimate (see Theorem 3.9). So suppose that $7C \cap \partial_p \Omega_T \neq \emptyset$. Then without loss of generality, we may assume in some appropriate coordinate system that $\Omega_T \supset Q_1^+(0, 2s^2)$, $S\Omega_T \supset \hat{T}_1(0, 2s^2)$ for some small $s > 0$ and that

$$\Omega_T \supset Q_1^+(0, 2s^2) \supset Q^+_{126r}(0, 2s^2) \supset Q^+_{18r}(0, 2s^2) \supset C_r \cap \Omega_T.$$

Now we apply Lemma 4.12 to the cube $Q^+_{18r}(0, 2s^2)$ when ε is replaced by $\varepsilon/(18)^{n+2}$, to obtain

$$\begin{aligned}
&\left|\left\{(x,t) \in \Omega_T : \mathcal{M}(|\nabla u|^2)(x,t) > N_1^2\right\} \cap C_r\right| \\
&\quad \leqslant \left|\left\{x \in Q_1^+(0,2s^2) : \mathcal{M}(|\nabla u|^2)(x,t) > N_1^2\right\} \cap Q^+_{18r}(0,2s^2)\right| \\
&\quad < \frac{\varepsilon}{18^{n+2}}\left|Q^+_{18r}\right| \\
&\quad = \varepsilon\left|C_r^+\right|,
\end{aligned}$$

which is a contradiction to (4.33). □

We take N_1, ε, and the corresponding δ given in Theorem 4.14. For the lateral boundary estimates we assume that $\Omega_T \supset Q_7^+(0,2)$ and that $S\Omega_T \supset \hat{T}_7(0,2)$.

Corollary 4.15. *Assume that A is uniformly parabolic and $(\delta, 1)$-vanishing. Suppose that $u \in \mathring{W}^{1,2}_*(\Omega_T)$ is a weak solution of (1.1), with the condition that*

$$\left|\left\{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u|^2)(x,t) > N_1^2\right\}\right| < \varepsilon\left|Q_1^+\right|. \tag{4.35}$$

Let k be a positive integer and set $\varepsilon_1 = 2(10)^{n+2}\varepsilon$. Then we have

$$\begin{aligned}
|\{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u|^2) > N_1^{2k}\}| \leqslant{}& \sum_{i=1}^{k} \varepsilon_1^i |\{(x,t) \in Q_1^+ : \mathcal{M}(|\mathbf{f}|^2) > \delta^2 N_1^{2(k-i)}\}| \\
&+ \varepsilon_1^k |\{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u|^2)) > 1\}|.
\end{aligned}$$

Proof. We prove by induction on k. Suppose first $k = 1$. Set

$$E = \{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u|^2) > N_1^2\}$$

and

$$F = \{(x,t) \in Q_1^+ : \mathcal{M}(|\mathbf{f}|^2) > \delta^2\} \cup \{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u|^2) > 1\}.$$

Then we know from (4.35), Theorems 4.14 and 2.6 that

$$|E| \leqslant \varepsilon_1 |F|,$$

and so our conclusion is true when $k = 1$.

Next assume that the conclusion is true for some positive integer k. Let us define $u_1 = \frac{u}{N_1}$ and corresponding $\mathbf{f}_1 = \frac{\mathbf{f}}{N_1}$. Then we see that u_1 is a weak solution of (1.1) and satisfies

$$|\{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u_1|^2)(x,t) > N_1^2\}| < \varepsilon |Q_1^+|.$$

By the induction hypothesis and from simple computations, we have

$$\begin{aligned}
&\left|\left\{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u|^2)(x,t) > N_1^{2(k+1)}\right\}\right| \\
&\quad = \left|\left\{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u_1|^2)(x,t) > N_1^{2k}\right\}\right| \\
&\quad \leqslant \sum_{i=1}^{k} \varepsilon_1^i \left|\left\{(x,t) \in Q_1^+ : \mathcal{M}(|\mathbf{f}_1|^2)(x,t) > \delta^2 N_1^{2(k-i)}\right\}\right| \\
&\qquad + \varepsilon_1^k \left|\left\{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u_1|^2)(x,t) > 1\right\}\right| \\
&\quad = \sum_{i=1}^{k} \varepsilon_1^i \left|\left\{(x,t) \in Q_1^+ : \mathcal{M}|\mathbf{f}|^2)(x,t) > \delta^2 N_1^{2(k+1-i)}\right\}\right| \\
&\qquad + \varepsilon_1^{k+1} |\{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u|^2)(x,t) > 1\}|.
\end{aligned}$$

These estimates in turn complete the induction on k.

Theorem 4.16. *Given $p > 2$, there is a small $\delta = \delta(p, n, \Lambda) > 0$ so that for all A with uniformly parabolic and $(\delta, 1)$-vanishing, and for all $\mathbf{f}$ with $\mathbf{f} \in L^p\left(Q_7^+(0,2)\right)$, if $u \in W_*^{1,2}\left(Q_7^+(0,2)\right)$ is a weak solution of*

$$\begin{cases} u_t - div(A\nabla u) = div\,\mathbf{f} & in \quad Q_7^+(0,2), \\ \qquad\qquad\quad u = 0 & on \quad \hat{T}_7(0,2), \end{cases} \tag{4.36}$$

then $\nabla u \in L^p(Q_1^+)$ with the estimate

$$\|\nabla u\|_{L^p(Q_1^+)} \leqslant C \|\mathbf{f}\|_{L^p(Q_7^+(0,2))}, \tag{4.37}$$

where the constant C is independent of u and $\mathbf{f}$.

Proof. As in the proof of Theorem 3.2, we may assume that

$$|\{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u|^2)(x,t) > N_1^2\}| < \varepsilon |C_1|$$

and

$$\sum_{k=0}^{\infty} N_1^{pk} |\{(x,t) \in Q_7^+(0,2) : \mathcal{M}(|\mathbf{f}|^2)(x,t) > \delta^2 N_1^{2k}\}| \leqslant 1. \tag{4.38}$$

In view of Corollary 4.15 and (4.38), an easy computation leads to

$$\sum_{k=0}^{\infty} N_1^{pk} \left|\left\{(x,t) \in Q_1^+ : \mathcal{M}(|\nabla u|^2)(x,t) > \delta^2 N_1^{2k}\right\}\right| \leqslant C \sum_{k=0}^{\infty} \left(N_1^p \varepsilon_1\right)^k < \infty$$

for some appropriate constant C depending only on δ, N_1^2, and p. We select an ε so that $N_1^p \varepsilon_1 < 1$ to obtain

$$\mathcal{M}\left(|\nabla u|^2\right) \in L^{p/2}(Q_1^+),$$

which in turn implies that

$$\nabla u \in L^p(Q_1^+)$$

with the estimate (4.37). □

Now we are ready to prove our lateral boundary estimate.

Theorem 4.17. *Given $p > 1$, there is a small $\delta = \delta(p, n, \Lambda) > 0$ so that for all A with A uniformly parabolic and $(\delta, 1)$-vanishing, and for all $\mathbf{f}$ with $\mathbf{f} \in L^p\left(Q_7^+(0,2)\right)$, if u is a weak solution of*

$$\begin{cases} u_t - div(A\nabla u) = div\,\mathbf{f} & in \quad Q_7^+(0,2), \\ \qquad\qquad u = 0 & on \quad \hat{T}_7(0,2), \end{cases} \tag{4.39}$$

then $u \in W_*^{1,p}(Q_1^+)$ with the estimate

$$\|u\|_{W_*^{1,p}(Q_1^+)} \leqslant C \|\mathbf{f}\|_{L^p(Q_7^+(0,2))} \tag{4.40}$$

for some constant C independent of u and $\mathbf{f}$.

Proof. The case $p = 2$ is classical and the case $1 < p < 2$ is recovered by duality argument. So we only consider the case $p > 2$. By the previous theorem and due to the fact that $u_t = div(A\nabla u + \mathbf{f})$ in $Q_7^+(0,2)$, we obtain

$$
\begin{aligned}
\|u\|_{W_*^{1,p}(Q_1^+)} &\leqslant \|u\|_{L^p(Q_1^+)} + \|\nabla u\|_{L^p(Q_1^+)} + \|A\nabla u + \mathbf{f}\|_{L^p(Q_1^+)} \\
&\leqslant C(\|u\|_{L^p(Q_7^+(0,2))} + \|\nabla u\|_{L^p(Q_7^+(0,2))} + \|\mathbf{f}\|_{L^p(Q_7^+(0,2))} \\
&\leqslant C\|\mathbf{f}\|_{L^p(Q_7^+(0,2))},
\end{aligned}
$$

which completes our proof. □

Remark 4.18. The estimates on the bottom and corner can be obtained by a zero extension of our weak solution along with the same approach we have used for the estimates on the lateral boundary.

5. Global $\mathring{W}_*^{1,p}$ regularity in Lipschitz domains with small Lipschitz constants

5.1. Flattening argument

In this section, we will show why we need the assumption that the boundary of the domain Ω is locally given by graphs with small Lipschitz constants. We first choose any point $x_0 \in \partial\Omega$. For our purpose, let us assume that

$$\Omega \cap B_r(x_0) = \left\{x \in B_r(x_0) : x_n > \gamma(x')\right\}$$

for some $r > 0$ and some $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ with $Lip(\gamma)$ small, where $Lip(\cdot)$ denotes the Lipschitz constant. Define then $y_i = x_i = \Phi^i(x)$ $(1 \leqslant i \leqslant n-1)$ and $y_n = x_n - \gamma(x') = \Phi^n(x)$ and write $y = \Phi(x)$. Now define $\Psi = \Phi^{-1}$ and write $x = \Psi(y)$. Choose $\rho > 0$ so small that B_ρ^+ lies in $\Phi(\Omega \cap B_r(x_0))$ and define $u_1(y,s) = u_1(\Psi(y),s)$ for all $y \in B_\rho^+$ and for all $s \in (-\rho^2, 0]$. If u is a weak solution of the PDE

$$u_t - div(A\nabla u) = div\,\mathbf{f} \quad \text{in} \quad \Omega_T,$$

then u_1 is a weak solution of

$$(u_1)_t - div(A_1\nabla u_1) = div\,\mathbf{f}_1 \quad \text{in} \quad Q_\rho^+,$$

where $\mathbf{f}_1(y,s) = \mathbf{f}(\Psi(y),s)$ and

$$A_1(y,s) = [\nabla\Phi(\Psi(y),s)]^T \cdot A(\Psi(y),s) \cdot [\nabla\Phi(\Psi(y),s)]. \tag{5.1}$$

A simple computation gives us

$$\nabla\Phi = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ . & . & \ldots & . \\ -\gamma_{x_1} & -\gamma_{x_2} & \ldots & 1 \end{bmatrix}.$$

Let us compute $[A_1]_{\mathrm{BMO}}$, assuming that $[A]_{\mathrm{BMO}}$ is small enough, where $[\cdot]_{\mathrm{BMO}}$ denotes the BMO semi-norm on $\mathbb{R}^{n+1}$ taken with respect to centered parabolic cubes. From an easy computation, we have

$$\|\nabla\Phi \cdot \nabla\Phi\|_\infty = n + \|\nabla\gamma\|_\infty^2. \tag{5.2}$$

Then from (5.1) and (5.2), it follows that

$$[A_1]_{\mathrm{BMO}} \leqslant C\left([A]_{\mathrm{BMO}} + Lip(\gamma)\right),$$

and so $[A_1]_{\mathrm{BMO}}$ is small provided $[A]_{\mathrm{BMO}} + Lip(\gamma)$ is small, which is our optimal regularity requirement in this work.

5.2. Proof of Theorem 1.5

We are finally all set to give the main result of this paper, Theorem 1.5.

Proof. Now that we have established the L^p estimates on the lateral boundary for the gradient of u in Q_1^+ in Theorem 4.17, we can proceed with the proof by standard scaling, covering and flattening arguments along with the interior estimates, boundary estimates on the corner and bottom, and duality argument. □

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